PHYS302 Fall 2023

Homework 6

1. Using matrix ray tracing, find a single matrix that represents a thick lens with radii of curvature R1 and R2 and thickness d. Show that the final matrix reproduces equation 6.2 in your text (the equation for the focal length of a thick lens).

$$S_{\text{thick lens}} = S_{\text{back interface}} \cdot S_{\text{propagation}} \cdot S_{\text{front interface}}$$

$$\rightarrow S_{\text{interface}} = \begin{bmatrix} 1 & \frac{n_1 - n_2}{R} \\ 0 & 1 \end{bmatrix} \qquad \rightarrow S_{\text{prop.}} = \begin{bmatrix} 1 & 0 \\ d/n & 1 \end{bmatrix}$$

$$S_{\text{thick lens}} = \begin{bmatrix} 1 & \frac{n_\ell - 1}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d/n_\ell & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1-n_\ell}{R_1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{n_\ell - 1}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1-n_\ell}{R_1} \\ d/n_\ell & \frac{d}{n_\ell}\left(\frac{1-n_\ell}{R_1}\right) + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{d}{n_\ell}\left(\frac{n_\ell - 1}{R_2}\right) & \frac{1-n_\ell}{R_1} + \frac{d}{n_\ell}\left(\frac{1-n_\ell}{R_1}\right)\left(\frac{n_\ell - 1}{R_2}\right) + \left(\frac{n_\ell - 1}{R_2}\right) \\ d/n_\ell & 1 - \frac{d}{n_\ell}\left(\frac{n_\ell - 1}{R_1}\right) \end{bmatrix}$$

From thin lens, we expect element in row 1 & column 2 to be $-1/f$.

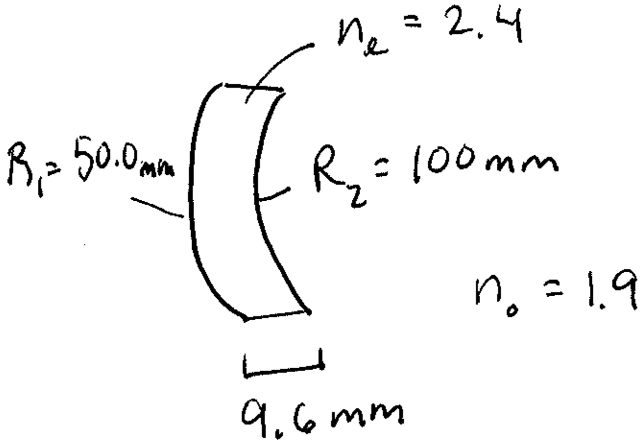
(The other elements relate to the finite thickness of the lens and its "principal planes.")

Element "B":

$$-\frac{1}{f} = \frac{-(n_e-1)}{R_1} + \frac{(n_e-1)}{R_2} - \frac{(n_e-1)}{R_1} \cdot \frac{(n_e-1)}{R_2} \cdot \frac{d}{n_e}$$

$$\boxed{\frac{1}{f} = (n_e-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{d(n_e-1)}{n_e R_1 R_2}\right]}$$

(2)



$$S_{thick\ lens} = \underbrace{\begin{bmatrix} 1 & \frac{n_o - n_e}{R_2} \\ 0 & 1 \end{bmatrix}}_{S_{int\,2}} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ d/n_e & 1 \end{bmatrix}}_{S_{prop}} \cdot \underbrace{\begin{bmatrix} 1 & \frac{n_e - n_o}{R_1} \\ 0 & 1 \end{bmatrix}}_{S_{int\,1}}$$

$$\rightarrow \frac{n_o - n_e}{R_2} = \frac{2.4 - 1.9}{100\ mm} = 0.005\ mm^{-1}$$

$$\rightarrow \frac{n_e - n_o}{R_1} = \frac{1.9 - 2.4}{50.0\,mm} = -0.01\ mm^{-1}$$

$$\rightarrow \frac{d}{n_e} = \frac{9.6\,mm}{2.4} = 4\ mm$$

$$S_{thick\ lens} = \begin{bmatrix} 1 & 0.005\ mm^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4\,mm & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.01\ mm^{-1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.02 & -0.0052\ mm^{-1} \\ 4.0\ mm & 0.96 \end{bmatrix}$$

③ $$S_{2\text{ traverses}} = S_{\substack{\text{reflection 2}\\ \text{(mirror 1)}}}\ S_{\text{prop. 2}}\ S_{\substack{\text{refl. 1}\\ \text{(mirror 2)}}}\ S_{\text{prop. 1}}$$

Recall: $S_{\text{mirror}} = \begin{bmatrix} -1 & \frac{-2n}{R} \\ 0 & 1 \end{bmatrix}$ & note that $R_2 = -r$ & $R_1 = r$

$$S_{2\text{ traverses}} = \begin{bmatrix} -1 & \frac{-2}{r} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix} \begin{bmatrix} -1 & -2/(-r) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$$

(d is negative when traveling backwards in cavity)

$$= \begin{bmatrix} -1 + \frac{2d}{r} & -2/r \\ -d & 1 \end{bmatrix} \begin{bmatrix} -1 + \frac{2d}{r} & 2/r \\ d & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{2d}{r} - 1\right)^2 - \frac{2d}{r} & \frac{2}{r}\left(\frac{2d}{r} - 1\right) - \frac{2}{r} \\ d - \frac{2d^2}{r} + d & 1 - 2d/r \end{bmatrix}$$

$$\rightarrow \frac{2}{r}\left(\frac{2d}{r} - 1\right) - \frac{2}{r} = \frac{4d}{r^2} - \frac{2}{r} - \frac{2}{r} = \frac{4d}{r^2} - \frac{4}{r} = \frac{4}{r}\left(\frac{d}{r} - 1\right)$$

$$= \begin{bmatrix} \left(\frac{2d}{r} - 1\right)^2 - \frac{2d}{r} & \frac{4}{r}\left(\frac{d}{r} - 1\right) \\ 2d\left(1 - \frac{d}{r}\right) & 1 - 2\frac{d}{r} \end{bmatrix}$$

(3, continued)

If $d = r$, $S_{2\ traverses} = \begin{bmatrix} (2\cdot 1 - 1)^2 - 2 & \frac{4}{r}(1-1) \\ 2d(1-1) & 1 - 2\cdot 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Four traverses is:

$$S_{2\ trav.} \cdot S_{2\ trav.} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ the identity matrix}$$

Thus, any ray, $\begin{bmatrix} a \\ b \end{bmatrix}$ multiplied by $S_{4\ traverses}$ will remain $\begin{bmatrix} r \\ b \end{bmatrix}$.

(4) From section 5.2.2:

$$l_o = \left[R^2 + (s_o+R)^2 - 2R(s+R)\cos\phi\right]^{1/2}$$

&

$$l_i = \left[R^2 + (s_i-R)^2 + 2R(s_i-R)\cos\phi\right]^{1/2}$$

Using $\cos\phi \approx 1 - \frac{\phi^2}{2}$:

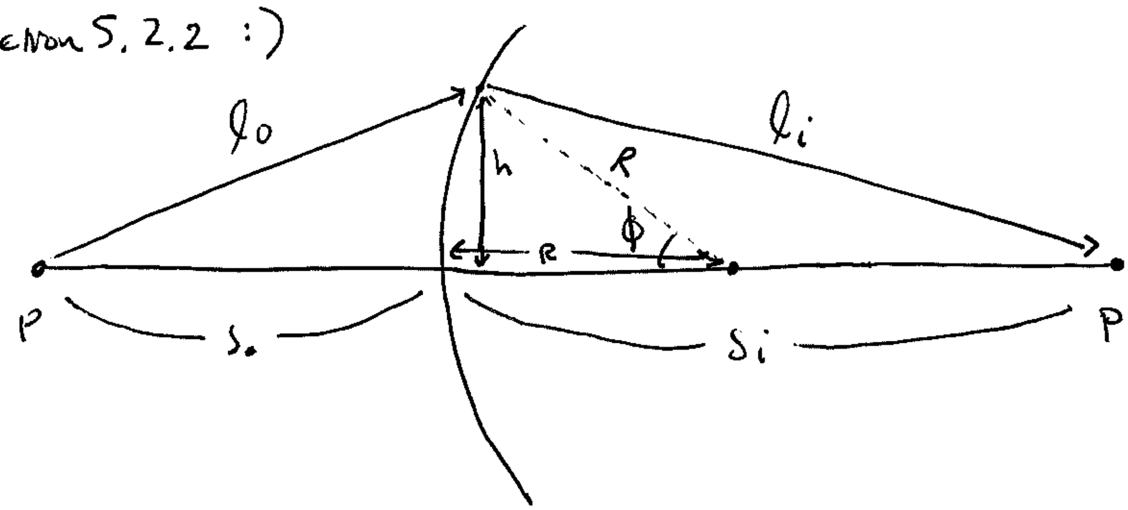
$$l_o \approx \sqrt{s_o^2 + R\phi^2(s_o+R)}$$

&

$$l_i \approx \sqrt{s_i^2 - R\phi^2(s_o+R)}$$

We want these expressions to depend on "h", the height (offset) of ray above the axis, not on "ϕ".

(From section 5.2.2:)



From this picture, we see that:

$$\sin \phi = \frac{h}{R} \approx \phi$$

(Note: why is this okay? we are letting ϕ^2 terms into our approximation as the "first order correction". However, $\sin \phi \approx \phi - \frac{\phi^3}{6} + \cdots$, so the next term in this expansion is $\phi^3 << \phi^2$ for smallish ϕ. This would only be a small correction on top of our correction.)

Let's write these expressions in the form we'll use!

$$\frac{1}{l_0} \approx \left(s_0^2 + R\left(\frac{h}{R}\right)^2 (s_0 + R)\right)^{-1/2}$$

$$\approx s_0^{-1}\left(1 + \frac{Rh^2}{R^2 s_0^2}(s_0 + R)\right)^{-1/2}$$

$$\approx s_0^{-1}\left[1 - \frac{1}{2}\frac{h^2}{R s_0^2}(s_0 + R)\right]$$

$$\boxed{l_0^{-1} = s_0^{-1} - \frac{(s_0 + R)\, h^2}{2 s_0^3 R}}$$

With similar math: $\boxed{l_i^{-1} = s_i^{-1} + \frac{(s_i - R)h^2}{2 s_i^3 R}}$

So, now our equation 5.5 reads:

(Left hand side:)

$$\frac{n_1}{l_o} + \frac{n_2}{l_i} = \frac{n_1}{s_o} - n_1 \frac{(s_o + R)h^2}{2 s_o^3 R} + \frac{n_2}{s_i} + n_2 \frac{(s_i - R)h^2}{2 s_i^3 R}$$

(Right hand side:)

$$\frac{1}{R}\left(\frac{n_2 s_i}{l_i} - \frac{n_1 s_o}{l_o}\right) = \frac{1}{R}\left[n_2 s_i \left(\frac{1}{s_i} + \frac{(s_i - R)h^2}{2 s_i^3 R}\right) - n_1 s_o \left(\frac{1}{s_o} - \frac{(s_o + R)h^2}{2 s_i^3 R}\right)\right]$$

$$= \frac{1}{R}\left[n_2 + \frac{n_2 (s_i - R)h^2}{2 s_i^3 R} - n_1 + \frac{n_1 (s_o + R)h^2}{2 s_o^3 R}\right]$$

(Together:)

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + \frac{n_1 (s_o + R)h^2}{2 s_o^3 R} + \frac{n_1 (s_o + R)h^2}{2 s_o^2 R^2} - \frac{n_2 (s_i - R)h^2}{2 s_i^3 R} + \frac{n_2 (s_i - R)h^2}{2 s_i^2 R^2}$$

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2 s_o}\left(\frac{s_o + R}{s_o^2 R} + \frac{s_o + R}{s_o R^2}\right) + \frac{n_2}{2 s_i}\left(\frac{s_i - R}{s_i R^2} - \frac{s_i - R}{s_i^2 R}\right)\right]$$

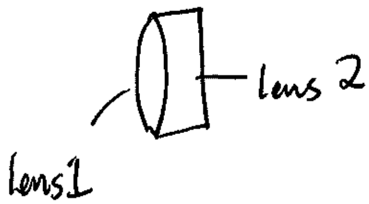
→ Note: $\frac{s_o}{s_o^2 R} + \frac{R}{s_o^2 R} + \frac{s_o}{s_o R^2} + \frac{R}{s_o R^2} = \frac{2}{s_o R} + \frac{1}{s_o^2} + \frac{1}{R^2} = \left(\frac{1}{s_o} + \frac{1}{R}\right)^2$

$\rightarrow$ similarly, $\left(\frac{s_i - R}{s_i R^2} - \frac{s_i - R}{s_i^2 R}\right) = \left(\frac{1}{R} - \frac{1}{s_i}\right)^2$

Putting it all together gets you equation 6.46:

$$\boxed{\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o} + \frac{1}{R}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]}$$

(5)



; $R_{front} = +30$ cm, $R_{middle} = -30$ cm, $R_{back} = \infty$

Using $\frac{1}{f} = (n_e - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$, we first find a focal length for each lens in all glasses at all the 2 wavelengths:

lens	glass	f_{red}	f_{blue}
1	crown	+ 29.19 cm	+ 28.68 cm
1	flint	(not required)	(not required)
2	crown	− 58.39 cm	− 57.35 cm
2	flint	− 48.86 cm	− 47.33 cm

Compound lens: $\frac{1}{f_{total}} = \frac{1}{f_1} + \frac{1}{f_2}$

In first part, lens 1 is crown glass and lens 2 is flint glass:

$f_{total\_red} = 72.51$ cm

$f_{total\_blue} = 72.78$ cm

$\rightarrow \boxed{\Delta f = 0.27 \text{ cm}}$

$\frac{\Delta f}{\bar{f}} \approx 0.0037 \rightarrow$ 0.37% change in f for the achromat

In second part, lens 1 is still crown glass, but lens 2 is flint glass:

$f_{total\_red} = 58.37$ cm

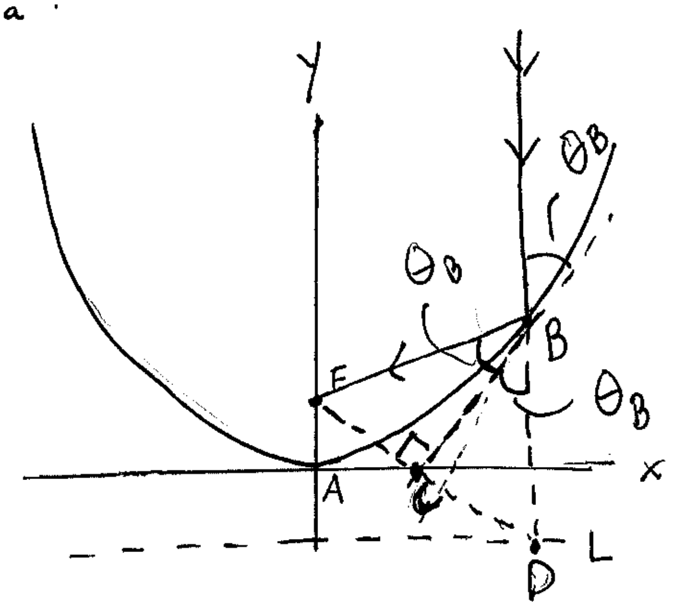
$f_{total\_blue} = 57.37$ cm

$\rightarrow \boxed{\Delta f = 1.00 \text{ cm}}$

$\frac{\Delta f}{\bar{f}} \approx 0.017 \rightarrow$ 1.7% change in f if not an achromat

Extra Credit:

parabola:



Definition of parabola: every point which is equidistant from point F (the focus) and line L (nearest pt.)

(1) From law of reflection, we know 3 angles labelled θ_B are the same.

(2) Triangles Δ FCB and Δ DCB must be identical. Thus, $\overline{BF} = \overline{BD}$ and D must be on line L iff F is the "focus" of parabola.

(3) This will be true for any choice of B (& θ_B). Thus, all parallel rays (to y-axis) hitting a point on the parabola will pass through point F after reflection.

Spherical vs. Parabolic mirror:

For small x, the surface of a spherical mirror is approximated by a parabola, and therefore the rays with small offset (x small) will focus at F:

Parabola equation: $y = \dfrac{x^2}{4f}$

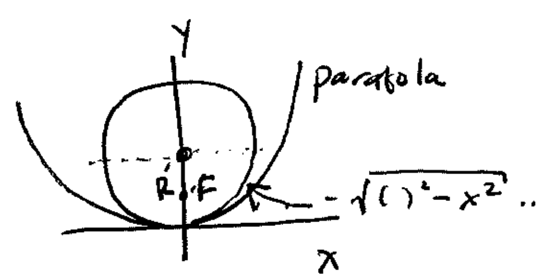
Circle equation (x-section of sphere): $x^2 + (y-R)^2 = R^2$

→ Remember $f = \left|-\dfrac{R}{2}\right|$

$$x^2 + (y-2f)^2 = (2f)^2$$

$$y = \pm\sqrt{(2f)^2 - x^2} + 2f$$

→ Choose negative square root ans. to get bottom half of circle:



$$y = 2f - 2f\sqrt{1 - \left(\frac{x}{2f}\right)^2}$$

Taylor expand square root:

$$\sqrt{1-x} \simeq 1 - \frac{x}{2} - \frac{x^2}{8} - \dots$$

(Circle:) $$y = 2f - 2f\left[1 - \frac{1}{2}\left(\frac{x}{2f}\right)^2 - \frac{1}{8}\left(\frac{x}{2f}\right)^4 - \cdots\right]$$

$$y = 2f - 2f + \frac{1}{2}\,\frac{x^2}{2f} + \frac{1}{8}\,\frac{x^4}{(2f)^3} + \cdots$$

$$y = \underbrace{\frac{x^2}{4f}}_{\text{parabola}} + \frac{x^4}{64 f^3} + \cdots$$

So, for $x^2 \ll 4f$, the spherical mirror looks parabolic. For $x^2 \approx 4f$, or more precisely, $\frac{x^4}{64 f^3}$ approaches a significant fraction of $\frac{x^2}{4f}$, the spherical aberrations will begin to shift the focus away from point F.